PARTISAN ADVANTAGE TRACKER at ippsr.msu.edu/PAT

Detailed Explanation of four notions of fairness.

Our calculation of Partisan Advantage depends on what we consider the *fair* way to turn votes for a party into seats won by that party. We illustrate the four *fairness rules* we consider with an example based on the representation of the State of Washington in Congress. According to the 2020 Census apportionment, the State of Washington is allocated 10 seats in the U.S. House of Representatives.

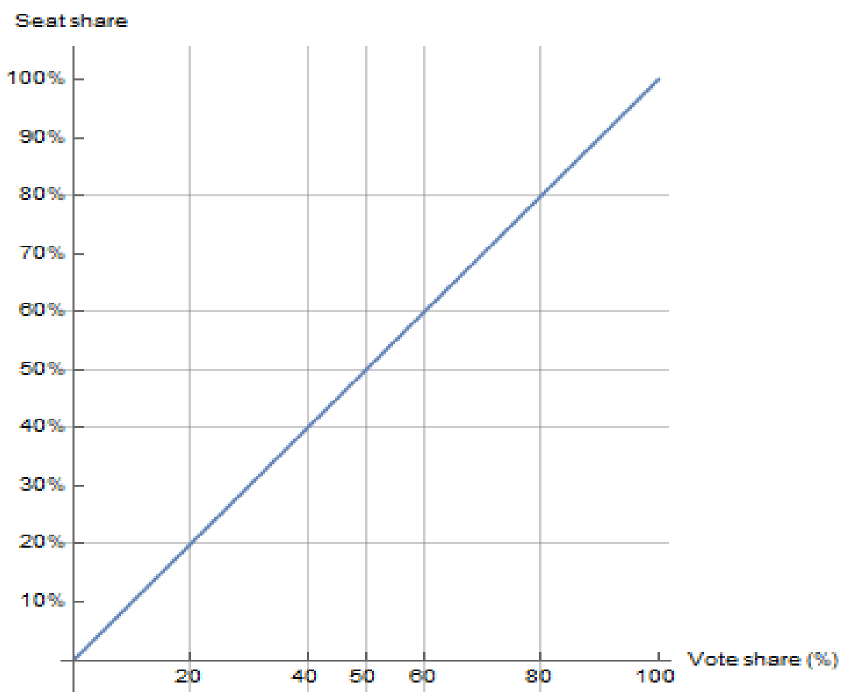
One possible notion of *fairness* is that a party's number of seats should be proportional to its *vote-share*. For example, if a political party wins 10% of the votes, under a proportional standard, the party should be awarded 10% of the seats –one seat out of ten. With 20% of the public's vote, two seats should be awarded, and so on, as in this table.

Vote-share	Proportional seats
10%	1
20%	2
…	…
100%	10

This can be written as a mathematical formula. Say "*v"* is the *vote-share* and "*s"* is the *seat-share*, then the proportional rule is

$$s = v$$

This formula can be represented graphically by the blue line in the graph below.



Proportional representation is encoded in the laws of many countries that elect many representatives per district, but under the U.S. electoral system, each congressional district is represented by a single member of the U.S. House of Representatives, whether this representative won 51% or 100% of the vote in the district. It follows that a party that wins a majority of the votes in every precinct of a state wins every seat in that state, regardless of how districts are drawn.

Proportional representation, in particular, is not a realistic standard for states with lopsided outcomes: just as President Ronald Reagan won 49 among 50 states with 58% of the national vote in the 1984 Presidential Election, in a state in which a party wins congressional elections by a large margin, this party typically wins most or all seats from that state, regardless of how the state's congressional districts are drawn.

We consider four notions of *fairness* recognizing this more-than-proportional feature of the U.S. electoral system. Each of these four notions yields a different benchmark.

1. Efficiency Gap Rule

This notion of *fairness* says that a party should win a *seat-share* majority equal to twice its *vote-share* majority. Thus, if a political party wins 55% of the vote, that party should win 60% of the seats; if the party wins 60% of the seats, it should logically be awarded 70% of the seats, and so on.

A motivation for this rule is the idea that the share of votes for each party that are not helping the party win any seat should be the same for both parties. These are votes cast for a losing candidate, or votes for a winning candidate in excess of the ones needed to win. A party with fewer of these votes that don't help win is more "efficient" at getting seats out of votes, and a possible idea of *fairness* is that both parties should be equally efficient, that is, their "Efficiency Gap" should be zero. In the special case in which turnout is (at least approximately) the same in every district, the rule such that a party wins a *seat-share* majority equal to twice its *vote-share* majority guarantees that the Efficiency Gap is (approximately) zero.

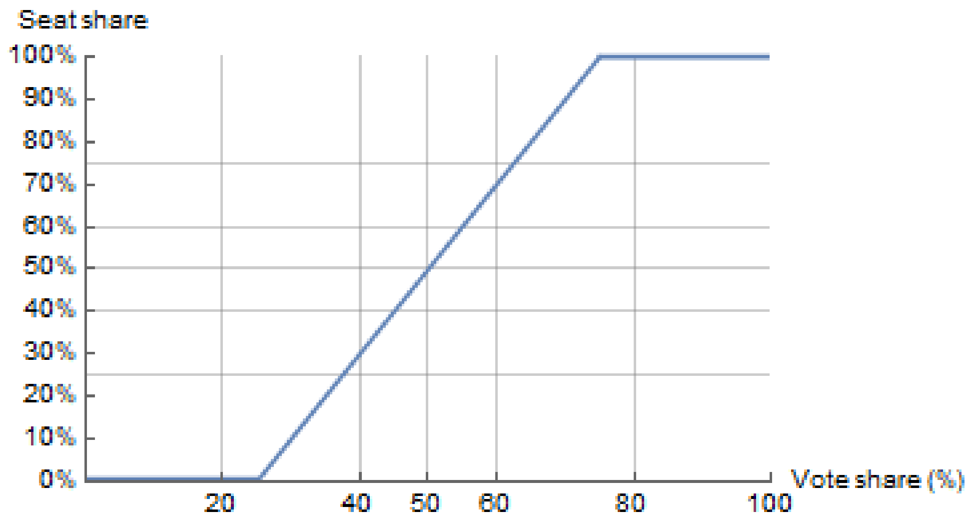
The following Table indicates the *Fair Seats* for a party under the Efficiency Gap Rule, depending on its *vote-share* in our example in the State of Washington.

Vote-share	Efficiency Gap seats
10%	0
25%	0
30%	1
40%	3
50%	5
60%	7
…	…
75%	10
100%	10

If vote-share v is between 0.25 and 0.75, the mathematical formula for this is:

$$s_{DB} = 2(v - 0.25)$$

Mathematically, this notion of *fairness* would have *seat-share* track the *vote-share* appear graphically as in the blue line below.



2. Quadratic Rule

One may object that the Efficiency Gap rule doesn't give any seats to parties that get 25% of the vote; whereas, gaining just an additional 1% *vote-share*, a party should suddenly earn a 2% *seat-share*. Why should increasing *vote-share* from 24% to 25% earn nothing, if increasing it from 25% to 26% earns a 2% *seat-share* gain? Wouldn't it be fairer for each of these similar *vote-share* gains to lead to a similar increase in *seat-share*?

Several normative considerations (detailed in the references listed under "Read more" below) suggest that vote-shares should translate to seat-shares according to a Quadratic function, which is smoother and curvier, without an obvious dog-leg. According to the notion of *fairness* given by this Quadratic rule, a party with less than 50% *vote-share* deserves *seat-share* equal to twice the square of its *vote-share*. A political party winning 40% of the vote would thus deserve 2*0.4*0.4=32% of *seat- share*.

Assuming we are in a two-party system, the majority party that wins more than 50% of the two-party vote then deserves the rest of the seats in the delegation, according to this rule. The table for our hypothetical Washington state result now looks like this

Vote share	Efficiency Gap seats	Quadratic rule seats
10%	0	0.2
25%	0	1.25
30%	1	1.8
40%	3	3.2

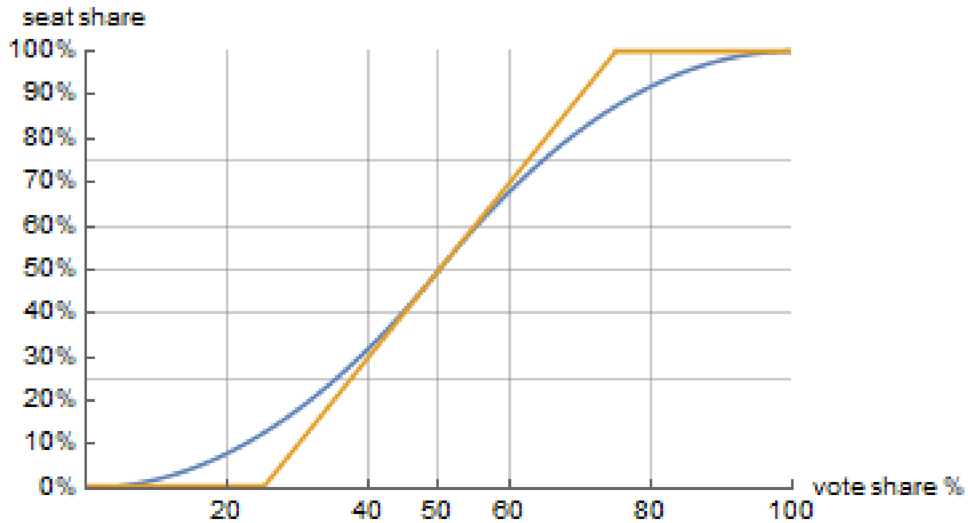
50%	5	5
60%	7	6.8
…	…	
75%	10	8.75
100%	10	10

We interpret fractional seats as averages. In this way, if there are five elections, “3.2” (as in the table above) means that it would be *fair* for one party to win 4 seats once and 3 seats four times over five elections.

If vote share *v* is below 0.5, the mathematical formula for the Quadratic Rule is

$$s_Q = 2v^2$$

Graphically, the Quadratic Rule notion of *fairness* shows a *seat-share* should track the *vote-share* like the blue line below (while the Efficiency Gap outcome appears in yellow). As you can see, the Quadratic Rule is flatter, requiring smaller parties to gain more representation than allowed by the Efficiency Gap Rule.



3. Cubic Rule

In practice, the actual pattern of how *vote-shares* turn into *seat-shares* follows not a straight line nor a quadratic curve, but looks more like a curve given by a mathematical cubic function. We can explain this function informally. Define the “*seat-share ratio*” as the *seat-share* of the largest party over the *seat-share* of the second-largest party, and similarly, define the “*vote-share ratio*” as the *vote-share* of the largest party over the *vote-share* of the second-largest party. Then, the cubic relation is that this *seat-share ratio* is typically close to the cube of their *vote-share ratio*.

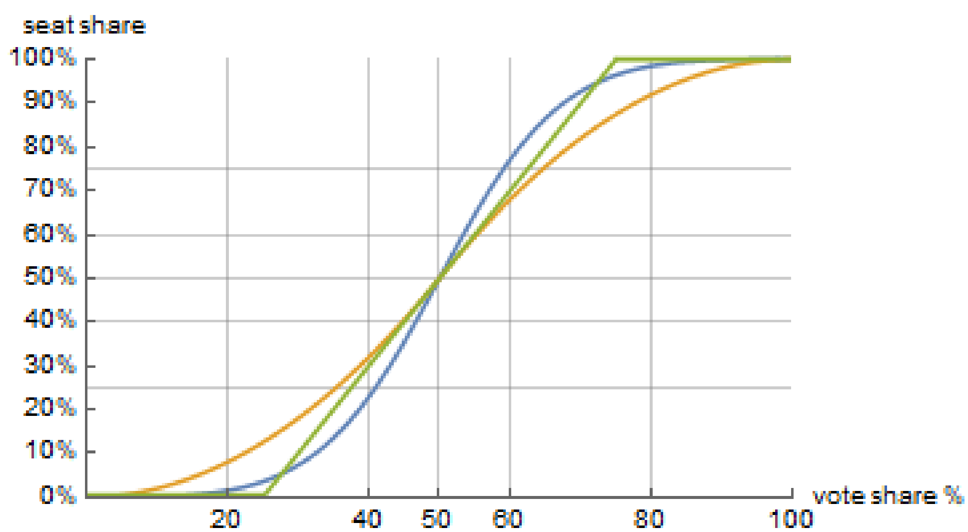
Mathematically, that means that according to the Cubic Rule, the *seat-share* follows the following formula:

$$s_{cube} = \frac{v^3}{3v^2 - 3v + 1}$$

The table for Washington state now looks like this:

Vote share	Efficiency Gap seats	Quadratic rule seats	Cubic rule seats
10%	0	0.2	0
25%	0	1.25	0.3
30%	1	1.8	0.7
40%	3	3.2	2.3
50%	5	5	5
60%	7	6.8	7.7
…	…		
75%	10	8.75	9.6
100%	10	10	10

And graphically, the Cubic Rule looks like the blue line below (with the Quadratic Rule in yellow, and the Efficiency Gap Rule in green).



4. Jurisdictional Rule

Because each member of the U.S. House of Representatives represents a particular geographic area (a Congressional District), <u>where</u> –and not just how many— votes for each party are cast matters to the election outcome and to its *fairness*. Under the jurisdictional notion of *fairness*, a political party should win seats in proportion to the number of people who live in jurisdictions (such as counties and cities) in which the party won more votes than any other party.

To be more precise, the jurisdictions we take into account (counties or cities) are those closest in population size to the population of one district. In most states, these are the counties, making the jurisdictional fair benchmark the number of seats proportional to the population in counties in which the party won the most seats. In states with counties with population greater than twice the population of a district (i.e. approximately greater than 1,600,000), we divide these larger counties, considering their largest cities in to their own jurisdiction (until the rest of the county has population below that of two districts).

As a result, according to this rule, the *fair* number of seats depends on where each party gets its votes. Thus, we cannot define the rule as a function of statewide vote-share. Rather, it is a function of which political party wins each jurisdiction. Back to State of Washington, for instance, in five hypothetical election results, --with winners by county (and the city of Seattle) as in the map in the left column, where jurisdictions won by Democrats are in blue, and those by Republicans are in red— the four *fair* rules give the following number of *fair* seats for the Democratic Party.

Map	Pop. in jurisdictions won by Dem	D Vote share	Efficiency Gap seats	Quadratic rule seats	Cubic rule seats	Jurisdictional rule seats
	9.6%	32.4%	1.8	2.1	0.9	1.0
	33.1%	48.9%	4.6	4.8	4.7	3.3
	64.3%	54.5%	5.9	5.8	6.3	6.4
	74.3%	59.9%	7	6.8	7.7	7.4
	100.0%	77.4%	10	9.0	9.8	10

In the first election, Democrats win only Seattle. In the last one, they win every jurisdiction. The fourth and fifth row are actual elections (2016 Washington gubernatorial, and 2020 U.S. presidential).

Read More

The following resources provide a more extensive analysis on each of the following topics:

i) How it is impossible to draw electoral district maps such that the number of seats and the number of votes for each party are proportional to each other:

-Duchin, M., Gladkova, T., Henninger-Voss, E., Klingensmith, B., Newman, H., & Wheelen, H. (2019). "Locating the representational baseline: Republicans in Massachusetts." Election Law Journal: Rules, Politics, and Policy, 18(4), 388-401.

ii) The Efficiency Gap Rule:

-*Stephanopoulos, Nicholas O., and Eric M. McGhee. "Partisan gerrymandering and the efficiency gap." U. Chi. L. Rev. 82 (2015): 831.*

-*McGhee, Eric. "Measuring efficiency in redistricting." Election Law Journal: Rules, Politics, and Policy 16.4 (2017): 417-442.*

iii) The Quadratic Rule and its motivations:

-*Barton, Jeffrey.* "*Fairness in Plurality Systems with Implications for Detecting Partisan Gerrymandering." Mathematical Social Sciences 117 (2022):* 69-90.

- Pegden, Wesley, Ariel D. Procaccia, and Dingli Yu. "A partisan districting protocol with provably nonpartisan outcomes." arXiv preprint arXiv:1710.08781 (2017).

iv) The Cubic Rule:

-Tufte, Edward R. "The relationship between seats and votes in two-party systems." American Political Science Review 67.2 (1973): 540-554.

v). The Jurisdictional Rule:

- Eguia, Jon X. *"A measure of partisan advantage in redistricting." Election Law Journal: Rules, Politics, and Policy 21.1 (2022): 84-103.*